Discrete Mathematics 121 (1993) 25–35
North-Holland

Long cycles, degree sums and neighborhood unions

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Received 20 October 1990
Revised 25 April 1991

Abstract

Broersma, H.J., J. van den Heuvel and H.J. Veldman, Long cycles, degree sums and neighborhood unions, Discrete Mathematics 121 (1993) 25–35.

For a graph G, define the parameters $\alpha(G)=\max\{|S|\mid S$ is an independent set of vertices of $G\}$, $\sigma_k(G)=\min\{\sum_{i=1}^k d(v_i)\mid\{v_1,\ldots,v_k\}$ is an independent set$\}$ and $NC_k(G)=\min\{|\bigcup_{i=1}^k N(v_i)|\mid\{v_1,\ldots,v_k\}$ is an independent set$\}$ $(k\geqslant 2)$. It is shown that every 1-tough graph G of order $n\geqslant 3$ with $\sigma_3(G)\geqslant n+r\geqslant n$ has a cycle of length at least $\min\{n, n+NC_{r+5+\varepsilon(n+r)}(G)-\alpha(G)\}$, where $\varepsilon(i)=3(\lceil\frac{1}{3}i\rceil-\frac{1}{3}i)$. This result extends previous results in Bauer et al. (1989/90), Faßbender (1992) and Flandrin et al. (1991). It is also shown that a 1-tough graph G of order $n\geqslant 3$ with $\sigma_3(G)\geqslant n+r\geqslant n$ has a cycle of length at least $\min\{n, 2NC_{\lfloor 1/8(n+6r+17)\rfloor}(G)\}$. Analogous results are established for 2-connected graphs.

1. Results

We use Bondy and Murty [6] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph of order n. The graph G is 1-*tough* if $\omega(G-S)\leqslant|S|$ for every subset S of $V(G)$ such that $\omega(G-S)>1$, where $\omega(G-S)$ denotes the number of components of $G-S$. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$ and the length of a longest cycle in G by $c(G)$. For $k\leqslant\alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $NC_k(G)$ the minimum cardinality of the neighborhood union of any k such vertices. For $k>\alpha(G)$ we set $\sigma_k(G)=k(n-\alpha(G))$ and $NC_k(G)=n-\alpha(G)$. Instead of $\sigma_1(G)$ and $NC_1(G)$ we use the more common notation $\delta(G)$. If no ambiguity can arise, we sometimes write α instead of $\alpha(G)$, etc.

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In Nash-Williams [10] it is proved that every 2-connected graph G of order n with $\delta(G)\geqslant\max\{\frac{1}{3}(n+2), \alpha(G)\}$ is hamiltonian. In [4], this result was extended as follows. (Note that $\sigma_3(G)\geqslant 3\delta(G)$ for any graph G.)

Theorem 1.1 (Bauer et al. [4]). *If G is a 2-connected graph of order n with $\sigma_3(G)\geqslant n+2$, then*

$$c(G)\geqslant\min\{n, n+\tfrac{1}{3}\sigma_3(G)-\alpha(G)\}.$$

Theorem 1.1 has a counterpart for 1-tough graphs.

Theorem 1.2 (Bauer et al. [4]). *If G is a 1-tough graph of order $n\geqslant 3$ with $\sigma_3(G)\geqslant n$, then*

$$c(G)\geqslant\min\,(n, n+\tfrac{1}{3}\sigma_3(G)-\alpha(G)\}.$$

Theorems 1.1 and 1.2 imply several known results. For details we refer to the surveys of Bauer et al. [1, 3].

Here we establish generalizations of Theorems 1.1 and 1.2, announced in [1], which also imply known results not contained in Theorems 1.1 and 1.2.

Define

$$\varepsilon(i):=\begin{cases}0 & \text{if } i\equiv 0 \pmod 3,\\ 2 & \text{if } i\equiv 1 \pmod 3,\\ 1 & \text{if } i\equiv 2 \pmod 3.\end{cases}$$

Theorem 1.3. *If G is a 2-connected graph of order n with $\sigma_3(G)\geqslant n+r\geqslant n+2$, then*

$$c(G)\geqslant\min\{n, n+NC_{r+2+\varepsilon(n+r)}(G)-\alpha(G)\}.$$

Theorem 1.4. *If G is a 1-tough graph of order $n\geqslant 3$ with $\sigma_3(G)\geqslant n+r\geqslant n$, then*

$$c(G)\geqslant\min\{n, n+NC_{r+5+\varepsilon(n+r)}(G)-\alpha(G)\}.$$

The proofs of Theorems 1.3 and 1.4 are postponed to Section 2.

Since clearly $NC_t(G)$ is a nondecreasing function of t and $NC_3(G)\geqslant\frac{1}{3}\sigma_3(G)$, Theorems 1.3 and 1.4 imply Theorems 1.1 and 1.2, respectively.

In Theorem 1.1 (Theorem 1.2), and hence in Theorem 1.3 (Theorem 1.4), the lower bound $n+2\,(n)$ imposed on $\sigma_3(G)$ cannot be relaxed without destroying the conclusion of the theorem, as shown by examples in [4]. Here we present examples showing that both the lower bound on $c(G)$ and the subscript of NC in the conclusion of Theorems 1.3 and 1.4 cannot be increased in general.

For $r\geqslant 2$, $p\geqslant r+2$ define the 2-connected graph $G_{p,r}$ as $K_p\vee((r+2)K_1+(p-r-1)K_2)$. Then, with $n=|V(G_{p,r})|$, we have $\sigma_3(G_{p,r})=3p=n+r$ and

$$c(G_{p,r})=n-1=n+NC_{r+2}(G_{p,r})-\alpha(G_{p,r})<n+NC_{r+3}(G_{p,r})-\alpha(G_{p,r})=n.$$

Hence, Theorem 1.3 is best possible if $n+r\equiv 0 \pmod 3$. The graphs obtained from $G_{p,r}$ by deleting one or two edges incident with an isolated vertex of $G_{p,r}-V(K_p)$ show that Theorem 1.3 is also best possible if $n+r\not\equiv 0 \pmod 3$.

Let F denote the unique graph with degree sequence $(1,1,1,3,3,3)$. For $r\geqslant 0$, $p\geqslant r+5$ define the 1-tough graph $H_{p,r}$ as $K_p \vee (F+(r+4)K_1+(p-r-5)K_2)$. Then with $n=|V(H_{p,r})|$, we have $\sigma_3(H_{p,r})=3p=n+r$ and

$$c(H_{p,r})=n-1=n+NC_{r+5}(H_{p,r})-\alpha(H_{p,r})<n+NC_{r+6}(H_{p,r})-\alpha(H_{p,r})=n.$$

Hence, Theorem 1.4 is best possible if $n+r\equiv 0 \pmod 3$. Again, by deleting one or two suitable edges of $H_{p,r}$ one obtains examples showing that Theorem 1.4 is best possible if $n+r\not\equiv 0 \pmod 3$.

Since clearly $NC_k(G)\leqslant n-\alpha(G)$ for any graph G of order n and any positive integer k, Theorems 1.3 and 1.4, respectively, have the following consequences.

Corollary 1.5. *If G is a 2-connected graph of order n with $\sigma_3(G)\geqslant n+r\geqslant n+2$, then*

$$c(G)\geqslant \min\{n, 2\,NC_{r+2+\varepsilon(n+r)}(G)\}.$$

Corollary 1.6. *If G is a 1-tough graph of order $n\geqslant 3$ with $\sigma_3(G)\geqslant n+r\geqslant n$, then*

$$c(G)\geqslant \min\{n, 2\,NC_{r+5+\varepsilon(n+r)}(G)\}.$$

Weaker versions of Corollaries 1.5 and 1.6, with the subscript of NC replaced by 2, were first established in [2].

Corollary 1.6 implies a recent result in [7], conjectured in [4].

Corollary 1.7 (Faßbender [7]). *If G is a 1-tough graph of order $n\geqslant 13$ with $\sigma_3(G)\geqslant \frac{1}{2}(3n-14)$, then G is hamiltonian.*

Proof. We apply Corollary 1.6 with $r=\lceil \frac{1}{2}(n-14)\rceil$. Suppose n is even, $n=2k$, say. Then $\varepsilon(n+r)=\varepsilon(3k-7)=1$, and hence $c\geqslant \min\{n, 2\,NC_{\frac{1}{2}n-1}\}$. Since G is 1-tough, $NC_{\frac{1}{2}n-1}\geqslant \frac{1}{2}n$. Thus, G is hamiltonian. A similar argument applies if n is odd. □

Corollary 1.7 extends the theorem in [9] that every 1-tough graph of order $n\geqslant 11$ with $\sigma_2(G)\geqslant n-4$ is hamiltonian. (Note that $\sigma_3(G)\geqslant \frac{3}{2}\sigma_2(G)$.)

Corollary 1.6 also implies a result in [8].

Corollary 1.8 (Flandrin et al. [8]). *If G is a 2-connected graph of order n such that $d(u)+d(v)+d(w)\geqslant n+|N(u)\cap N(v)\cap N(w)|$ for every independent set $\{u,v,w\}$, then G is hamiltonian.*

Proof. Let G satisfy the stated conditions. It is easily seen that G is 1-tough [1]. We show that $NC_3\geqslant \frac{1}{2}n$. The proof is then completed by applying Corollary 1.6 with $r=0$ and observing that $NC_5\geqslant NC_3$. By definition of NC_3, we are done if $\alpha<3$. Hence, assume $\alpha \geqslant 3$ and let $\{u_1, u_2, u_3\}$ be an independent set with $|\bigcup_{i=1}^{3} N(u_i)| = NC_3$. Denote by n_i the number of vertices of G adjacent to exactly i of the vertices u_1, u_2, u_3 $(i = 1, 2, 3)$. Then

$$n + n_3 \leqslant \sum_{i=1}^{3} d(u_i) = n_1 + 2n_2 + 3n_3,$$

implying that

$$n \leqslant n_1 + 2n_2 + 2n_3 \leqslant 2(n_1 + n_2 + n_3) = 2\left|\bigcup_{i=1}^{3} N(u_i)\right| = 2NC_3. \qquad \square$$

Summarizing we have shown that Theorem 1.4 is a common generalization of Theorem 1.2 and Corollaries 1.7 and 1.8. We note that Theorem 1.2 contains neither Corollary 1.7 nor Corollary 1.8. Examples of graphs which are hamiltonian by Corollary 1.7, but not by Theorem 1.2, can be found among spanning subgraphs of $K_p \vee pK_1$ $(p \geqslant 7)$. Examples of graphs which are hamiltonian by Corollary 1.8, but not by Theorem 1.2, occur in [1].

We conclude by showing that Corollaries 1.5 and 1.6 admit (partial) improvements.

As in Theorems 1.1 and 1.3, the lower bound $n+2$ imposed on $\sigma_3(G)$ in Corollary 1.5 cannot be relaxed. Also, the conclusion of Corollary 1.5 is sharp in the sense that longer cycles are not implied by the hypothesis, as shown by suitable complete bipartite graphs. However, in contrast with the situation for Theorem 1.3, the subscript of NC in Corollary 1.5 can be improved for certain values of r. Our next result shows that $r + 2 + \varepsilon(n+r)$ can be replaced by $\lfloor \frac{1}{8}(n + 6r + 17) \rfloor$, which yields an improvement if $r \leqslant \frac{1}{2}n - 7$. Note that if G satisfies the hypothesis of Corollary 1.5 with $r \geqslant \frac{1}{2}n - 1$, then G is hamiltonian, since

$$2NC_{r+2+\varepsilon(n+r)}(G) \geqslant 2NC_3(G) \geqslant \tfrac{2}{3}\sigma_3(G) \geqslant \tfrac{2}{3}(\tfrac{3}{2}n - 1) > n - 1.$$

Theorem 1.9. *If G is a 2-connected graph of order n with $\sigma_3(G) \geqslant n + r \geqslant n + 2$ and $n \geqslant 8t - 6r - 17$, then*

$$c(G) \geqslant \min\{n, 2NC_t(G)\}.$$

A combined proof of Theorems 1.9 and 1.11 is given in Section 2. We believe that Theorem 1.9 admits further improvement.

Conjecture 1.10. If G is a 2-connected graph of order n with $\sigma_3(G) \geqslant n + r \geqslant n + 2$ and $n \geqslant 6t - 4r - 11$, then

$$c(G) \geqslant \min\{n, 2NC_t(G)\}.$$

Conjecture 1.10 would, in a sense, be best possible: if $n = |V(G_{p,r})|$, $p + r$ is even and $t = \frac{1}{2}(p + r) + 2$, then $n = 6t - 4r - 12$, but $c(G_{p,r}) = n - 1 < 2NC_t(G_{p,r}) = n$.

The subscript of NC in Corollary 1.6 can also be replaced by $\lfloor \frac{1}{8}(n + 6r + 17) \rfloor$, which yields an improvement if $r \leqslant \frac{1}{2}n - 19$ (cf. Corollary 1.7).

Theorem 1.11. *If G is a 1-tough graph of order $n \geqslant 3$ with $\sigma_3(G) \geqslant n+r \geqslant n$ and $n \geqslant 8t-6r-17$, then*

$$c(G) \geqslant \min\{n, 2\,NC_t(G)\}.$$

As in Theorems 1.2 and 1.4, the lower bound n imposed on $\sigma_3(G)$ in Corollary 1.6 and Theorem 1.11 cannot be relaxed. With respect to Theorem 1.11, we not only believe that the subscript of NC admits further improvement, but also that the lower bound on $c(G)$ is not sharp (cf. [2, Conjecture 27]).

Conjecture 1.12. If G is a 1-tough graph of order $n \geqslant 3$ with $\sigma_3(G) \geqslant n+r \geqslant n$ and $n \geqslant 6t-4r-11$, then

$$c(G) \geqslant \min\{n, 2\,NC_t(G)+4\}.$$

As Conjecture 1.10, Conjecture 1.12 would be best possible: if $n=|V(H_{p,r})|$, $p+r$ is even and $t=\frac{1}{2}(p+r)+2$, then $n=6t-4r-12$, but $c(H_{p,r})=n-1<2\,NC_t(H_{p,r})+4=n$. Also, Conjecture 1.12 would be another generalization of Corollary 1.7.

2. Proofs of the main results

We need some additional terminology and notation.

Let G be a graph. If $v \in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v. If $S \subseteq V(G)$, then $N(S)=\bigcup_{v\in S} N(v)$.

Let C be a cycle of G and $u, v \in V(C)$. We denote by $\vec{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. By $u\vec{C}v$ we denote the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. (The same vertices, in reverse order, are given by $v\overleftarrow{C}u$.) We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $S \subseteq V(C)$, then $S^+=\{v^+ \mid v \in S\}$ and $S^-=\{v^- \mid v \in S\}$. If $V(G)-V(C)$ is an independent set, then C is called a *dominating cycle* of G. If C is not a Hamilton cycle of G, then $\mu(C)$ denotes $\max\{d(v) \mid v \in V(G)-V(C)\}$. If C is a longest cycle of G not containing the vertex v, then C is called a *v-cycle*.

The following lemmas are of use in more than one of the proofs of the main results. Part (a) of our first lemma is a central lemma in [4].

Lemma 2.1. *Let G be a graph of order n with $\delta(G) \geqslant 2$ and $\sigma_3(G) \geqslant n$. Assume G contains a longest cycle $\vec{C}$ which is a dominating cycle, and a vertex $x \in V(G)-V(C)$. Then:*

(a) *$(V(G)-V(C)) \cup N(x)^+$ is an independent set, and*

(b) *if $u \in V(C)$, $xu^+ \in E(G)$ and $xu^- \in E(G)$, then $N(u) \subseteq V(C)$ and $(V(G)-V(C)) \cup N(u)^+$ is an independent set.*

Proof of (b). Since $u \in N(x)^+$, $N(u) \subseteq V(C)$ by (a). To see that $(V(G)-V(C)) \cup N(u)^+$ is an independent set, apply (a) with $\vec{C}$ and x replaced by $xu^+\vec{C}u^-x$ and u, respectively. □

The first part of the next lemma is a result in [5], the second part is implicit in the proof of [4, Theorem 10].

Lemma 2.2. *Let G be a 2-connected graph of order n with $\sigma_3(G) \geqslant n+r \geqslant n+2$. Then every longest cycle is a dominating cycle. Moreover, if G is nonhamiltonian, G contains a longest cycle C with $\mu(C) \geqslant \frac{1}{3}(n+r)$.*

The first part of our third lemma is [4, Theorem 5], the second part is implicit in the proof of [4, Theorem 9].

Lemma 2.3. *Let G be a 1-tough graph of order n with $\sigma_3(G) \geqslant n+r \geqslant n \geqslant 3$. Then every longest cycle is a dominating cycle. Moreover, if G is nonhamiltonian, G contains a longest cycle C with $\mu(C) \geqslant \frac{1}{3}(n+r)$.*

We are now ready to prove the main results. In the proofs a longest cycle C of G is considered. We will say that a property $\mathscr{P}$ of G holds *by* (*lca*) (longest cycle argument) if the contrary to $\mathscr{P}$ implies the existence of a cycle C' longer than C. (lca) will often represent an argument which is standard in hamiltonian graph theory. Sometimes the cycle C' will be given between brackets after the statement of $\mathscr{P}$. We start with (an outline of) the proof of Theorem 1.4.

Proof of Theorem 1.4. Let G be a nonhamiltonian 1-tough graph of order n with $\sigma_3(G) \geqslant n+r \geqslant n \geqslant 3$, $\vec{C}$ a longest cycle of G with a fixed orientation for which $\mu(C)$ is maximal, and u_0 a vertex in $V(G)-V(C)$ with $d(u_0)=\mu(C)$. By Lemma 2.3, C is a dominating cycle and $d(u_0) \geqslant \frac{1}{3}(n+r)$. Set $A=N(u_0)$, $k=d(u_0)$ and let $v_1, \ldots, v_k$ be the vertices of A, occurring on $\vec{C}$ in consecutive order. For $i=1,\ldots,k$ set $u_i=v_i^+$ and $w_i=v_{i+1}^-$ (indices mod k). The set $u_i\vec{C}w_i$ will be called a *segment*; $u_i\vec{C}w_i$ is a *p-segment* if $|u_i\vec{C}w_i|=p$. Let s be the number of 1-segments. Since $k > \frac{1}{3}|V(C)|$, $s \geqslant 1$. Let $u_{i_1}, \ldots, u_{i_s}$ be the vertices of the 1-segments and set $S=\{u_{i_1}, \ldots, u_{i_s}\}$. Without loss of generality we assume $d(u_{i_1}) \geqslant d(u_{i_2}) \geqslant \cdots \geqslant d(u_{i_s})$ and $i_1=1$. We abbreviate $\varepsilon(n+r)$ to ε.

We state some observations, each followed by a proof, which will be used repeatedly.

(1) $(V(G)-V(C)) \cup A^+ \cup N(S)^+$ is an independent set.

Apply Lemma 2.1 and (lca) (several times).

(2) $|V(C)| \geqslant 3k-s \geqslant n+r-s$.

C contains s 1-segments and hence $k-s$ t-segments with $t \geqslant 2$. Thus, $|V(C)| \geqslant 2s+3(k-s)=3k-s$. Since $k \geqslant \frac{1}{3}(n+r)$, $3k-s \geqslant n+r-s$.

(3) Let x_1 and x_2 be two nonadjacent vertices in $V(C)-A$. If G contains an x_1-cycle and an x_2-cycle, then $n+r-2k\leqslant d(x_i)\leqslant k$ $(i=1,2)$.

The choice of C and u_0 implies $d(x_i)\leqslant k$ $(i=1,2)$. The rest of (3) follows from the inequalities $d(x_1)\leqslant k$, $d(x_2)\leqslant k$ and $d(u_0)+d(x_1)+d(x_2)\geqslant \sigma_3\geqslant n+r$.

(4) Assume $N(u_{i_p})=A$ and $u_iw_j\in E(G)$, where $i\neq j$. Then $u_{i_p}\in w_j\vec{C}u_i, d(w_i)\geqslant n+r-2k$, $w_iv_{i_p}\notin E(G)$ and $w_iv_{i_p+1}\notin E(G)$.

By (lca), $u_{i_p}\in w_j\vec{C}u_i$ $(v_{i_p}u_0v_{j+1}\vec{C}v_iu_{i_p}\vec{C}w_ju_i\vec{C}v_{i_p})$. Since $\{u_0,u_{i_p},w_i\}$ is an independent set, $d(w_i)\geqslant\sigma_3-d(u_0)-d(u_{i_p})=\sigma_3-2k\geqslant n+r-2k$. By (lca), $w_iv_{i_p}\notin E(G)$ $(u_{i_p}\vec{C}v_iu_0v_{j+1}\vec{C}v_{i_p}w_i\vec{C}u_iw_j\vec{C}v_{i+1}u_{i_p})$ and $w_iv_{i_p+1}\notin E(G)$ $(u_{i_p}v_i\vec{C}v_{i_p+1}w_i\vec{C}u_iw_j\vec{C}v_{i+1}u_0v_{j+1}\vec{C}u_{i_p})$.

(5) Assume $N(u_1)=A$ and $u_iw_j\in E(G)$, where $i\neq j$ and i is chosen minimal. Then $N(w_i)\cap(A^+\cup A^-)\subseteq\{u_i\}$.

By (lca), $N(w_i)\cap A^-=\emptyset$. The minimality of i implies $N(w_i)\cap A^+\cap u_1\vec{C}v_i=\emptyset$. By (4), $N(w_i)\cap A^+\cap u_i\vec{C}v_1\subseteq\{u_i\}$.

(6) Let $u_i\vec{C}w_i$ be a t-segment with $t\geqslant 3$ and assume $u_{i_p}u_i^+\in E(G)$. Then $N(u_i)\cap A^-=N(u_i^{++})\cap A^+=\emptyset$.

Apply (lca).

A subset X of $V(G)$ will be called *suitable* if $|X|\geqslant r+5+\varepsilon$, $N(X)\subseteq V(C)$ and both X and $(V(G)-V(C))\cup N(X)^+$ are independent sets. If a suitable set X exists, then we are done, since

$$\alpha\geqslant|V(G)|-|V(C)|+|N(X)^+|=|V(G)|-|V(C)|+|N(X)|$$
$$\geqslant n-|V(C)|+NC_{|X|},$$

and hence $|V(C)|\geqslant n+NC_{r+5+\varepsilon}-\alpha$. We will distinguish several cases, in each of which we either exhibit a suitable set or reach a contradiction. By (2), $n-1\geqslant|V(C)|\geqslant n+r-s$, so $s\geqslant r+1$.

Case 1: $s=r+1$. By (2), $|V(C)|=n-1$ and $k=\frac{1}{3}(n+r)$. Apart from the 1-segments, C contains 2-segments only. Since G is 1-tough, $\omega(G-A)\leqslant|A|$. Hence, there exist i and j with $i\neq j$ such that $u_iw_j\in E(G)$. Assume i is chosen minimal. Since G contains a w_i-cycle $(u_iw_j\vec{C}v_{i+1}u_0v_{j+1}\vec{C}u_i)$ and a u_1-cycle $(v_1u_0v_2\vec{C}v_1)$, by (3) we have $k=n+r-2k\leqslant d(u_1)\leqslant k$. Hence, $N(u_1)=A$. By (4) and (5), $d(w_i)\geqslant k$ and $N(w_i)\subseteq(A-\{v_1,v_2\})\cup\{u_i\}$, whence $d(w_i)\leqslant k-1$, a contradiction.

Case 2: $s=r+2$. By (2), there are three possibilities.

Case 2.1: $|V(C)|=n-1$, $k=\frac{1}{3}(n+r)$. Then $\varepsilon=0$. Apart from the 1- and 2-segments, C contains exactly one 3-segment, say $u_lu_l^+w_l$. We distinguish two subcases.

Case 2.1.1: $u_{i_p}u_l^+\in E(G)$ for some $p\in\{1,\dots,s\}$. Then $N(S)\subseteq A\cup\{u_l^+\}$. By (6), $N(u_l), N(w_l)\subseteq A\cup\{u_l^+\}$ also. Using (1) we conclude that the set $S\cup\{u_0,u_l,w_l\}$, of cardinality $s+3=r+5+\varepsilon$, is suitable.

Case 2.1.2: $u_{i_p}u_l^+ \notin E(G)$ for all $p\in\{1,\dots,s\}$. From (3) (with $x_1, x_2\in S$) and the fact that $n+r-2k=k$ we deduce that $N(u_{i_q})=A$ for $q=1,\dots,s$. If $u_l w_l\in E(G)$, then by (lca), $N(u_l^+)\cap(A^+\cup A^-)=\{u_l, w_l\}$. Since $\omega(G-A)\leqslant|A|$, there exist i and j with $i\neq j$ such that $u_i w_j\in E(G)$. This is also true if $u_l w_l\notin E(G)$, otherwise $\omega(G-(A\cup\{u_l^+\}))>|A\cup\{u_l^+\}|$. Choose i minimal. By (4) and (5), $d(w_i)\geqslant k$ and $N(w_i)\subseteq(A-\{v_1, v_2, v_{i_2}, v_{i_2+1}\})\cup\{u_i, u_l^+\}$. Since $|\{v_1, v_2, v_{i_2}, v_{i_2+1}\}|\geqslant 3$, we reach a contradiction.

Case 2.2: $|V(C)|=n-1$, $k=\frac{1}{3}(n+r+1)$. Apart from the 1-segments, C contains 2-segments only. By (3), $k-1\leqslant d(u_1)\leqslant k$ and $k-1\leqslant d(u_{i_2})\leqslant k$. Since $d(u_0)+d(u_1)+d(u_{i_2})\geqslant\sigma_3\geqslant n+r=3k-1$ and $d(u_1)\geqslant d(u_{i_2})$, it follows that $d(u_1)=k$, and hence $N(u_1)=A$. Since G is 1-tough, there exists a smallest i such that $u_i w_j\in E(G)$ for some $j\neq i$. By (4) and (5), $N(w_i)\subseteq(A-\{v_1, v_2\})\cup\{u_i\}$, so $d(w_i)\leqslant k-1$. Now consider the greatest h such that $u_g w_h\in E(G)$ for some $g\neq h$. By (4), $i\leqslant g<h$. By (4) and (5) (now applied to $\overleftarrow{C}$), $N(u_h)\subseteq(A-\{v_1, v_2\})\cup\{w_h\}$, so $d(u_h)\leqslant k-1$. But then $n+r\leqslant\sigma_3\leqslant d(u_0)+d(u_h)+d(w_i)\leqslant 3k-2=n+r-1$, a contradiction.

Case 2.3: $|V(C)|=n-2$, $k=\frac{1}{3}(n+r)$. Let $V(G)-V(C)=\{u_0, x\}$. Apart from the 1-segments, C contains 2-segments only. By Lemma 2.1(a) (applied to both $\overrightarrow{C}$ and $\overleftarrow{C}$), $N(x)\subseteq A$ and we reach a contradiction as in case 1.

Case 3: $s=r+3$. By (2), there are six possibilities.

Case 3.1: $|V(C)|=n-1$, $k=\frac{1}{3}(n+r)$. Then $\varepsilon=0$. Apart from the 1- and 2-segments, C contains either one 4-segment or two 3-segments.

Case 3.1.1: C contains one 4-segment, say $u_l u_l^+ w_l^- w_l$.

Case 3.1.1.1: $u_{i_p}u_l^+\in E(G)$ or $u_{i_p}w_l^-\in E(G)$ for some $p\in\{1,\dots,s\}$. We may assume $u_{i_p}u_l^+\in E(G)$. Set $X=S\cup\{u_0, u_l\}$. Then $|X|=s+2=r+5+\varepsilon$. By (6), $N(X)\subseteq A\cup\{u_l^+\}$ and $N(X)^+$ is an independent set. Hence, X is a suitable set.

Case 3.1.1.2: $u_{i_p}u_l^+\notin E(G)$ and $u_{i_p}w_l^-\notin E(G)$ for all $p\in\{1,\dots,s\}$. As in case 2.1.2, $N(u_{i_q})=A$ for $q=1,\dots,s$.

Case 3.1.1.2.1: $u_i w_j\in E(G)$ for some i, j, $i\neq j$. Choose i minimal. By (4), $d(w_i)\geqslant k$. By (4) and (5), $N(w_i)\subseteq(A-\{v_1, v_2, v_{i_2}, v_{i_2+1}, v_{i_3}, v_{i_3+1}\})\cup\{u_i, u_l^+, w_l^-\}$, whence $d(w_i)\leqslant k-1$, a contradiction.

Case 3.1.1.2.2: $u_i w_j\notin E(G)$ whenever $i\neq j$. Since G is 1-tough, some 2-segment contains a vertex adjacent to u_l^+ or w_l^-. We assume $w_l^- u_i\in E(G)$ for some $i\neq l$; the other possibilities can be handled similarly. Suppose $w_l u_l^+\in E(G)$. Then by (lca), $N(w_i)\subseteq(A-\{v_1, v_2\})\cup\{u_i\}$ (cf. (4)). On the other hand, $d(w_i)\geqslant\sigma_3-d(u_0)-d(u_1)\geqslant n+r-2k=k$. This contradiction shows that $w_l u_l^+\notin E(G)$. Also, by (lca), $w_l u_l\notin E(G)$. Now $S\cup\{u_0, w_l\}$ is a suitable set.

Case 3.1.2: C contains two 3-segments, say $u_l u_l^+ w_l$ and $u_m u_m^+ w_m$ ($l<m$).

Case 3.1.2.1: $u_{i_p}u_l^+\in E(G)$ or $u_{i_p}u_m^+\in E(G)$ for some $p\in\{1,\dots,s\}$. We may assume $u_{i_p}u_l^+\in E(G)$. Set $X=S\cup\{u_0, u_l\}$. By (6), $N(X)\subseteq A\cup\{u_l^+, u_m^+\}$ and, if $u_m^+\notin N(X)$, $N(X)^+$ is an independent set. Hence, X is a suitable set if $u_m^+\notin N(X)$. Now assume $u_m^+\in N(X)$. If $u_m^+\in N(S)$, then $N(w_m)\cap A^+=\emptyset$ by (6). If $u_m^+\in N(u_l)$, then $N(w_m)\cap A^+=\emptyset$ by (lca). Hence, $A^+\cup\{w_l, w_m\}$ is an independent set, implying that again X is suitable.

Case 3.1.2.2: $u_{i_p}u_l^+\notin E(G)$ and $u_{i_p}u_m^+\notin E(G)$ for all $p\in\{1,\dots,s\}$, As in case 2.1.2, $N(u_{i_q})=A$ for $q=1,\dots,s$. Let $t\in\{l,m\}$. If $N(u_t^+)\cap(A^+\cup A^-)\neq\{u_t, w_t\}$, then

$u_t w_t \notin E(G)$ by (lca). Since $\omega(G-Y) \leqslant |Y|$ for $Y = A$, $A \cup \{u_l^+\}$, $A \cup \{u_m^+\}$, $A \cup \{u_l^+, u_m^+\}$, it follows that $u_i w_j \in E(G)$ for some i, j, $i \neq j$. Choose i minimal. By (4) and (5), $d(w_i) \geqslant k$ and $N(w_i) \subseteq (A - \{v_1, v_2, v_{i_2}, v_{i_2+1}, v_{i_3}, v_{i_3+1}\}) \cup \{u_i, u_l^+, u_m^+\}$, a contradiction.

Case 3.2: $|V(C)| = n-1$, $k = \frac{1}{3}(n+r+1)$. Then $\varepsilon = 1$. Apart from the 1- and 2-segments, C contains one 3-segment, say $u_l u_l^+ w_l$.

Case 3.2.1: $u_{i_p} u_l^+ \in E(G)$ for some $p \in \{1, \ldots, s\}$. As in case 2.1.1, the set $S \cup \{u_0, u_l, w_l\}$, of cardinality $s+3 = r+6 = r+5+\varepsilon$, is suitable.

Case 3.2.2: $u_{i_p} u_l^+ \notin E(G)$ for all $p \in \{1, \ldots, s\}$. Arguing as in case 2.2, now applying (3) with $\{x_1, x_2\} = \{u_{i_2}, u_{i_3}\}$, we obtain $N(u_{i_2}) = A$ and hence $N(u_1) = A$. As in case 2.1.2, there exsists a smallest i such that $u_i w_j \in E(G)$ for some $j \neq i$. By (4) and (5), $N(w_i) \subseteq (A - \{v_1, v_2, v_{i_2}, v_{i_2+1}\}) \cup \{u_i, u_l^+\}$, so $d(w_i) \leqslant k-1$. Defining h as in case 2.2 we similarly obtain $d(u_h) \leqslant k-1$, and reach a contradiction as in case 2.2.

Case 3.3: $|V(C)| = n-1, k = \frac{1}{3}(n+r+2)$. Apart from the 1-segments, C contains 2-segments only. Since $3d(u_1) \geqslant d(u_1) + d(u_{i_2}) + d(u_{i_3}) \geqslant \sigma_3 \geqslant n+r = 3k-2$, $d(u_1) \geqslant k$, and hence $N(u_1) = A$. Since G is 1-tough, there exists a smallest i such that $u_i w_j \in E(G)$ for some $j \neq i$. By (4) and (5), $N(w_i) \subseteq (A - \{v_1, v_2\}) \cup \{u_i\}$, so $d(w_i) \leqslant k-1$. With h defined as in case 2.2, we also have $d(u_h) \leqslant k-1$. Now $d(u_{i_2}) \geqslant \sigma_3 - d(w_i) - d(u_h) \geqslant n+r-2k+2 = k$, so $N(u_{i_2}) = A$. But then, in fact, $N(w_i) \subseteq (A - \{v_1, v_2, v_{i_2}, v_{i_2+1}\}) \cup \{u_i\}$. Hence, $d(w_i) \leqslant k-2$ and, similarly, $d(u_h) \leqslant k-2$. We reach the contradiction $n+r \leqslant \sigma_3 \leqslant d(u_0) + d(u_h) + d(w_i) \leqslant 3k-4 = n+r-2$.

Case 3.4: $|V(C)| = n-2$, $k = \frac{1}{3}(n+r)$. Let $V(G) - V(C) = \{u_0, x\}$. Apart from the 1- and 2-segments, C contains one 3-segment, say $u_l u_l^+ w_l$. By Lemma 2.1 (a), $N(x) \subseteq A \cup \{u_l^+\}$. As in case 2.1, we obtain a suitable set or a contradiction.

Case 3.5: $|V(C)| = n-2, k = \frac{1}{3}(n+r+1)$. Let $V(G) - V(C) = \{u_0, x\}$. Then $N(x) \subseteq A$ and we reach a contradiction as in case 2.2.

Case 3.6: $|V(C)| = n-3$, $k = \frac{1}{3}(n+r)$. Let $V(G) - V(C) = \{u_0, x, y\}$. Then $N(x)$, $N(y) \subseteq A$ and we reach a contradiction as in case 1.

Case 4: $s \geqslant r+4$. If $s \geqslant r+6$, then the set $S \cup \{u_0\}$, of cardinality $s+1 \geqslant r+7 \geqslant r+5+\varepsilon$, is suitable. The remaining cases are similar to previous cases. Since no new arguments are required, we omit the details.

Proof of Theorem 1.3. Let G be a nonhamiltonian 2-connected graph of order n with $\sigma_3(G) \geqslant n+r \geqslant n+2$. Copy the notation and terminology used in the proof of Theorem 1.4, with $r+5+\varepsilon$ replaced by $r+2+\varepsilon$ in the definition of a suitable set. Then in all possible cases, $S \cup \{u_0\}$ is a suitable set. □

Theorems 1.9 and 1.11 are established by combining Lemmas 2.2 and 2.3, respectively, with the following.

Lemma 2.4. *Let G be a graph of order n with $\delta(G) \geqslant 2$ and $\sigma_3(G) \geqslant n+r \geqslant n$. Assume G contains a longest cycle C such that C is a dominating cycle and $\mu(C) \geqslant \frac{1}{3}(n+r)$. If $n \geqslant 8t - 6r - 17$, then $c(G) \geqslant \min\{n, 2\,NC_t(G)\}$.*

Proof. Let G be a nonhamiltonian graph satisfying the stated conditions. Among all longest cycles which are dominating cycles, let $\vec{C}$ be one for which $\mu(C)$ is maximum. Define u_0, A, k and, for $i=1,\dots,k$, v_i, u_i, w_i as in the proof of Theorem 1.4. Then $d(u_0)\geqslant\frac{1}{3}(n+r)$. Set $T_i=u_i\vec{C}w_i (i=1,\dots,k)$. Again T_i is called a segment (p-segment if $|T_i|=p$), while S and s denote, respectively, the set of 1-segments and its cardinality. Since $k>\frac{1}{3}|V(C)|$, $s\geqslant 1$. Assume without loss of generality that $u_1\in S$ and u_1 has maximum degree among all vertices of S.

Set

$$S'=\{x\in V(C)\,|\,x^+,x^-\in N(u_1)\},$$

$$S''=\{x\in V(C)\,|\,x^+\in N(u_0),\ x^-\in N(u_1)\},$$

$$S_1=S\cup S'\cup S'' \text{ and } s_1=|S_1|.$$

We distinguish two cases.

Case 1: $t\leqslant s_1+1$. Then we can find a subset X of $V(G)$ of cardinality t consisting of u_0 and $t-1$ vertices in S_1. By Lemma 2.1, $N(X)\subseteq V(C)$. By (several applications of) (lca), X is an independent set and $N(X)\cap N(X)^+=\emptyset$. Hence, $c\geqslant 2|N(X)|\geqslant 2NC_t$.

Case 2: $t\geqslant s_1+2$. Since $d(u_0)=k\geqslant\frac{1}{3}(n+r)$, we have

$$n-1\geqslant c\geqslant 2s+3(k-s)\geqslant n+r-s,$$

implying that $s\geqslant r+1$ and, since $s_1\geqslant s$, $t\geqslant r+3$. Hence, by the hypothesis of the lemma,

$$(7)\quad k\geqslant\tfrac{1}{3}(n+r)\geqslant\tfrac{1}{3}(8t-5r-17)\geqslant\tfrac{1}{3}(3t-2)>t-1.$$

Set $q=t-s_1-1$ and let $T_{i_1},\dots,T_{i_q}$ be the first q segments following T_1 on C which contain no vertex of S_1. The existence of these segments is guaranteed by (7). Set $W=\{w_{i_1},\dots,w_{i_q}\}$ and $X=S_1\cup W\cup\{u_0\}$. Then X is an independent set of cardinality t with $N(X)\subseteq V(C)$. We are done if $c\geqslant 2|N(X)|$, so assume

$$(8)\quad |N(X)|\geqslant\tfrac{1}{2}(c+1).$$

We will derive a contradiction from (8). Set $Y=N(W)\cap A^+\cap v_1\vec{C}v_{i_q+1}$ and $Z=N(W)\cap A^+\cap v_{i_q+1}\vec{C}v_1$. Then $N(X)\subseteq(V(C)-(A^+\cup A^-))\cup Y\cup Z$, so

$$\begin{aligned}(9)\qquad |N(X)|&\leqslant c-s-2(k-s)+|Y|+|Z|\\&\leqslant c-s-2(k-s)+(t-1-s)+|Z|\\&=c-2k+t-1+|Z|.\end{aligned}$$

Let H be a component of $C-(S_1\cup S_1^-\cup Z\cup Z^+\cup Z^-)$. If $v\in N(u_1)\cap V(H)$, then v^+, $v^{++}\in V(H)-N(u_1)$. Observing that the sets S_1, S_1^-, Z, Z^+, Z^- are pairwise disjoint and $N(u_1)\cap(Z\cup Z^+\cup Z^-)=\emptyset$, we conclude that

$$\begin{aligned}(10)\qquad d(u_1)&\leqslant|S_1^-|+\tfrac{1}{3}(c-|S_1\cup S_1^-\cup Z\cup Z^+\cup Z^-|)\\&=s_1+\tfrac{1}{3}(c-2s_1-3|Z|)=\tfrac{1}{3}c+\tfrac{1}{3}s_1-|Z|.\end{aligned}$$

Summing (9) and (10), we obtain

$$d(u_1)+|N(X)|\leqslant\tfrac{4}{3}c+\tfrac{1}{3}s_1-2k+t-1.$$

Hence, by (8),

(11) $d(u_1)\leqslant\frac{5}{6}c+\frac{1}{3}s_1-2k+t-\frac{3}{2}$.

We now show that

(12) $\sigma_3\leqslant 2d(u_0)+d(u_1)$.

If C contains at least two 1-segments, then, by the way C, u_0 and u_1 were chosen, $\sigma_3\leqslant d(u_0)+2d(u_1)\leqslant 2d(u_0)+d(u_1)$. Hence, assume T_1 is the only 1-segment. Then $\sigma_3=n$, $c=n-1$, $d(u_0)=\frac{1}{3}n$ and all segments other than T_1 are 2-segments. There exist i and j with $i,j\neq 1$ and $i\neq j$ such that $u_iw_j\in E(G)$, otherwise $|N(X)|=|N(u_0)|+q=\frac{1}{3}n+t-2\leqslant\frac{1}{3}n+\frac{1}{8}(n+17)-2=\frac{1}{24}(11n+3)$ and we contradict (8). Thus, G has a w_i-cycle. By the choice of C and u_0, $d(w_i)\leqslant d(u_0)$. Since $d(u_0)+d(u_1)+d(w_i)\geqslant n$ and $d(u_1)\leqslant d(u_0)$, it follows that $d(u_0)=d(u_1)=d(w_i)=\frac{1}{3}n$ and (12) holds with equality.

From (11), (12) and the hypothesis of case 2, we obtain

$$\begin{aligned}n+r\leqslant\sigma_3&\leqslant 2\,d(u_0)+d(u_1)=2k+d(u_1)\\&\leqslant\tfrac{5}{6}\,c+\tfrac{1}{3}s_1+t-\tfrac{3}{2}\\&\leqslant\tfrac{5}{6}\,(n-1)+\tfrac{1}{3}(t-2)+t-\tfrac{3}{2},\end{aligned}$$

whence $n\leqslant 8t-6r-18$, contradicting the hypothesis of the lemma. □

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